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Real hypersurfaces in the complex quadric with parallel normal Jacobi operator

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First we introduce the notion of parallel normal Jacobi operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. Next we give a complete classification of real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ with parallel normal Jacobi operator.

1 Introduction

In a class of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [12], [13], and [14]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2U_m)$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Suh [15] and [16]). The complex quadric also can be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [4]). Accordingly, the complex quadric admits both a complex conjugation structure A and a Kähler structure J, which anti-commutes with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [3] and Reckziegel [10]).

In the complex projective space $\mathbb{C}P^m$, a full classification with isometric Reeb flow was obtained by Okumura in [5]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$. In the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$, Suh [12], [13] has given the classification when the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric. Moreover, in [14] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

Recently, Suh [12] proved a non-existence property for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor, and in [13] gave a characterization by harmonic curvature for a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In view of the previous many results a natural expectation might be that the classification involves at least the totally geodesic $Q^{m-1} \subset Q^m$. But, surprisingly, for real hypersurfaces in Q^m the situation is quite different from the results mentioned above. As an example of such situations, in [15] and [16] we have introduced the following result:

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Theorem 1 *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

On the other hand, in [8,9] we have introduced the notion of structure Jacobi operator R_ξ, which is a symmetric operator for real hypersurfaces in the complex projective space $\mathbb{C}P^m$, and have used it to study some principal curvatures for a tube over a totally geodesic submanifold (see Berger [1], Klein [3] and Reckziegel [10]). From such a view point, recently, Jeong, Kim and Suh [2] have investigated real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M, and Pérez, Jeong and Suh [7] generalize this notion to recurrent normal Jacobi operator, that is, $(\nabla_X \bar{R}_N)Y = \beta(X)\bar{R}_N Y$ for any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$, where β denotes a certain recurrent 1-form defined on M. Moreover, Pak, Suh and Woo [6] have focused on the study of commuting Jacobi operators, and Pérez and Santos [8], Pérez, Santos and Suh [9] respectively have investigated recurrent structure Jacobi operator $\nabla_X R_\xi = \beta(X) R_\xi$ or Lie ξ-parallel structure Jacobi operator in $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$ for any vector field X on a hypersurface M in $\mathbb{C}P^m$.

When we consider a hypersurface M in the complex quadric Q^m, the unit normal vector field N of M in Q^m can satisfy two conditions: N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [15, 16] and [17]). In the first case where M has an $\mathfrak{A}$-isotropic unit normal N, we have introduced in [15] and [16] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. Moreover, this kind of tube is characterized by Suh [15] and [16] in terms of Reeb parallel or Reeb invariant shape operator of M in Q^m respectively. In the second case where N is $\mathfrak{A}$-principal we have proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m.

In a real hypersurface M in the complex quadric Q^m we introduce the notion of parallel normal Jacobi operator $\bar{R}_N$ that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M. This has a geometric meaning that the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are *parallel*, that is, *invariant* under any parallel displacements along any curves on M in Q^m. Moreover, this meaning gives that if Γ is an eigenspace of the normal Jacobi operator $\bar{R}_N$, then for any $X \in \Gamma$ we have $\nabla_Y X \in \Gamma$ along any direction Y on M in Q^m.

In this paper, with this kind of geometric notion in the complex quadric Q^m, we prove the following

Main Theorem. *There do not exist any real hypersurfaces in the complex quadric Q^m, $m \geq 3$, with parallel normal Jacobi operator.*

2 The complex quadric

For more background to this section we refer to [4], [10], [15], and [17]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini–Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini–Study metric $\bar{g}$ is defined by $\bar{g}(X, Y) = \Phi(JX, Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1, 1)$-form Φ given by $\Phi = -4i\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \cdots, z^{m+1}] \in \mathbb{C}P^{m+1} \mid z^j \neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z^k}{z^j}$ for $j, k = 0, \ldots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m.

For each $z \in Q^m$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$. The tangent space $T_z Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\bar{z})$ of $\mathbb{C}z \oplus \mathbb{C}\bar{z}$ in $\mathbb{C}^{m+2}$, where $\bar{z} \in \nu_z Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point z (see Kobayashi and Nomizu [4]).

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to $S^2 \times S^2$ the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector $\bar{z}$ of Q^m at a point $z \in Q^m$ we denote by $A = A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to $\bar{z}$. It is defined by

$$A_{\bar{z}} w = -\bar{\nabla}_w \bar{z} = -\bar{w}$$

for any tangent vector field $w \in T_z Q^m$, $z \in Q^m$. Here the Levi–Civita connection $\bar{\nabla}$ of the complex projective space $\mathbb{C}P^{m+1}$ is induced from the Euclidean connection $\tilde{\nabla}$ of the complex Euclidean space $\mathbb{C}^{m+2}$. Then the shape operator becomes an involution on the tangent space $T_z Q^m$, that is, $A_{\bar{z}}^2 = I$ as follows:

$$A_{\bar{z}}^2 w = A_{\bar{z}} A_{\bar{z}} w = -A_{\bar{z}} \bar{w} = \bar{\nabla}_{\bar{w}} \bar{z} = \bar{\bar{w}} = w$$

for any tangent vector field w belonging to $T_z Q^m$, $z \in Q^m$ and the tangent space of $T_z Q^m$ is decomposed as

$$T_z Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}})$ is the $+1$-eigenspace and $JV(A_{\bar{z}})$ is the (-1)-eigenspace of $A_{\bar{z}}$. Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_z Q^m$, or equivalently, is a complex conjugation on $T_z Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $z \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at z. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at z of such a reflection is a conjugation on $T_z Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_z Q^m$ corresponds to the family of real forms S^m of Q^m containing z, and the subspaces $V(A) \subset T_z Q^m$ correspond to the tangent spaces $T_z S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY\end{aligned}$$

where X, Y and Z are vector fields belonging to $T_z Q^m$, $z \in Q^m$.

Recall that a nonzero tangent vector $W \in T_z Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

For every unit tangent vector $W \in T_z Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3 Some general equations

Let M be a real hypersurface in Q^m. When we consider a transform JX of the Kaehler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M, where ϕX denotes a tangential component of JX, $\xi = -JN$, and $\eta(X) = g(\xi, X)$. Then it naturally satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \text{and} \quad \eta(\xi) = 1.$$

In this case we call (ϕ, ξ, η, g) the induced *almost contact metric structure*. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma as follows:

Lemma 3.1 *(See [15].) For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M. Then the equation of Codazzi is given as follows:

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1), we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ & = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ & = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see [7, Proposition 3]). Since the Reeb vector field ξ is given by $\xi = -JN$, we have

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

This implies that $g(\xi, AN) = 0$.

4 A key lemma

Let M be a real hypersurface in a complex quadric Q^m which is a complex hypersurface in $\mathbb{C}P^{m+1}$, and let N be a unit normal vector field to M in Q^m. From the Riemannian curvature tensor $\bar{R}$ of the complex quadric Q^m given in Section 2, the normal Jacobi operator $\bar{R}_N$ is defined in such a way that

$$\begin{aligned}\bar{R}_N(X) &= \bar{R}(X,N)N \\ &= X + g(JN,N)JX - g(JX,N)JN - 2g(JX,N)JN \\ &\quad + g(AN,N)AX - g(AX,N)AN + g(JAN,N)JAX - g(JAX,N)JAN\end{aligned}$$

for any tangent vector field X in T_zM and the unit normal N of M in T_zQ^m, $z \in Q^m$. Then the normal Jacobi operator $\bar{R}_N$ becomes a symmetric operator on the tangent space T_zM, $z \in M$, of Q^m. From this, by the complex structure J and the complex conjugations $A \in \mathfrak{A}$, together with the fact that $g(A\xi, N) = 0$ and $\xi = -JN$ in Section 3, the normal Jacobi operator $\bar{R}_N$ is given by

$$\begin{aligned}\bar{R}_N(Y) &= Y + 3\eta(Y)\xi + g(AN,N)AY \\ &\quad - g(AY,N)AN - g(AY,\xi)A\xi\end{aligned}$$

for any $Y \in T_zM$, $z \in M$. Then the covariant derivative of $\bar{R}_N$ is given by

$$(\nabla_X \bar{R}_N)Y = \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y). \tag{4.1}$$

Now let us put $Y = \xi$ in above equation. Then it follows that

$$\bar{R}_N(\xi) = 4\xi + \{g(AN,N) - g(A\xi,\xi)\}A\xi = 4\xi + 2g(AN,N)A\xi,$$

where we have used that $g(A\xi,\xi) = g(AJN,JN) = -g(JAN,JN) = -g(AN,N)$.

Here we use the assumption of parallel normal Jacobi operator. Then (4.1) gives that

$$\begin{aligned}0 &= (\nabla_X \bar{R}_N)\xi \\ &= \nabla_X(\bar{R}_N(\xi)) - \bar{R}_N(\nabla_X\xi) \\ &= 4\nabla_X\xi + 2\big\{g(\bar{\nabla}_X AN, N) + g(AN, \bar{\nabla}_X N)\big\}A\xi + 2g(AN,N)\nabla_X(A\xi) \\ &\quad - \Big\{\nabla_X\xi + g(AN,N)A\nabla_X\xi - g(A\nabla_X\xi, N)AN - g(A\nabla_X\xi,\xi)A\xi\Big\}\end{aligned} \tag{4.2}$$

From this, by taking an inner product with the unit normal N, we have

$$g(AN,N)\{q(X)g(JA\xi,N) + g(A\phi SX,N) + g(SX,\xi)g(AN,N) - g(SX,A\xi)\} = 0.$$

Then by putting $X = \xi$, we get

$$g(AN,N)\{q(\xi)g(A\xi,\xi) - 2\alpha g(A\xi,\xi)\} = 0. \tag{4.3}$$

On the other hand, we know that

$$\begin{aligned}\bar{\nabla}_X(AY) - A\nabla_X Y &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - A\nabla_X Y \\ &= q(X)JAY + A\sigma(X,Y) \\ &= q(X)JAY + g(SX,Y)AN,\end{aligned}$$

where q denotes an 1-form defined on T_zQ^m, $z \in Q^m$. So naturally it follows that

$$\begin{aligned}(\nabla_X A)\xi &= \bar{\nabla}_X(A\xi) - A\nabla_X\xi \\ &= (\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi - A\nabla_X\xi \\ &= q(X)JA\xi + g(SX,\xi)AN.\end{aligned}$$

From this, together with (4.1), it follows that

$$\begin{aligned}0 = &\ (\nabla_X \bar{R}_N)Y \\ = &\ 3g(\phi SX, Y)\xi + 3\eta(Y)\phi SX \\ &+\{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}AY \\ &+g(AN, N)\{q(X)JAY + g(SX, Y)AN\} \\ &-\{q(X)g(JAY, N) + g(SX, Y)g(AN, N)\}AN \\ &+g(AY, SX)AN - g(AY, N)\{(\bar{\nabla}_X A)N + A\bar{\nabla}_X N\} \\ &-g((\bar{\nabla}_X A)Y, \xi)A\xi - g(AY, \phi SX + \sigma(X, \xi))A\xi \\ &-g(AY, \xi)\{(\nabla_X A)\xi + A\nabla_X \xi\},\end{aligned} \tag{4.4}$$

where we have used the equation of Gauss $\bar{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi)$, $\sigma(X, \xi)$ denotes the normal bundle $T^\perp M$ valued second fundament tensor on M in Q^m. From this, putting $Y = \xi$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$, and $\bar{\nabla}_X N = -SX$ we have

$$\begin{aligned}0 = &\ 3\phi SX \\ &+\{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}A\xi \\ &+g(AN, N)\{q(X)JA\xi + g(SX, \xi)AN\} \\ &-\{q(X)g(JA\xi, N) + g(SX, \xi)g(AN, N)\}AN \\ &+g(A\xi, SX)AN - g(q(X)JA\xi, \xi)A\xi \\ &-g(A\xi, \phi SX + \sigma(X, \xi))A\xi \\ &-g(A\xi, \xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)AN\}.\end{aligned} \tag{4.5}$$

From this, by taking the inner product with the unit normal N, we have

$$\begin{aligned}&g(A\xi, SX)g(AN, N) - q(X)g(A\xi, \xi)g(JA\xi, N) \\ &\quad -g(A\xi, \xi)g(A\phi SX, N) - g(A\xi, \xi)g(SX, \xi)g(AN, N) = 0.\end{aligned} \tag{4.6}$$

Then by putting $X = \xi$ and using the assumption of Hopf, we have

$$q(\xi)g(A\xi, \xi)^2 = 0. \tag{4.7}$$

This gives that $q(\xi) = 0$ or $g(A\xi, \xi) = 0$. The latter case implies that the unit normal N is $\mathfrak{A}$-isotropic. When the first case $q(\xi) = 0$ holds, (4.3) implies

$$\alpha g(A\xi, \xi)g(AN, N) = 0. \tag{4.8}$$

Then from (4.8) we can assert the following lemma

Lemma 4.1 *Let M be a Hopf real hypersurface in complex quadric Q^m, $m \geq 3$, with parallel normal Jacobi operator. Then the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Proof. When the Reeb function α is non-vanishing, (4.8) gives $g(A\xi, \xi) = 0$ or $g(AN, N) = 0$. Then by (3.3), we know that $\cos^2 t - \sin^2 t = \cos 2t = 0$. This gives that $t = \frac{\pi}{4}$. Then by also (3.3), the unit normal N becomes $\mathfrak{A}$-isotropic, that is, $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$ for some $Z_1, Z_2 \in V(A)$.

When the Reeb function α identically vanishes, let us show that N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. In order to do this, from the condition of Hopf, we can differentiate $S\xi = \alpha\xi$ and use the equation of Codazzi (3.1) in Section 3, then we get the formula

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

From this, if we put $\alpha = 0$, together with the fact $g(\xi, AN) = 0$ in Section 3, we know $g(Y, AN)g(\xi, A\xi) = 0$ for any $Y \in T_z M$, $z \in M$. This gives that the vector AN is normal, that is, $AN = g(AN, N)N$ or $g(A\xi, \xi) = 0$, which implies respectively the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. This completes the proof of our lemma. □

By virtue of this lemma, we distinguish between two classes of real hypersurfaces in complex quadric Q^m with parallel normal Jacobi operator: those that have $\mathfrak{A}$-principal unit normal, and those that have $\mathfrak{A}$-isotropic unit normal vector field N. We treat the respective cases in Sections 5 and 6.

5 Parallel normal Jacobi operator with $\mathfrak{A}$-principal normal

In this section let us consider a real hypersurface M in a complex quadric with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$.

Then the normal Jacobi operator $\bar{R}_N$ in Section 4 becomes

$$\bar{R}_N(X) = X + 3\eta(X)\xi + AX - \eta(X)\xi = X + 2\eta(X)\xi + AX, \tag{5.1}$$

where we have used that $AN = N$ and

$$\begin{aligned} g(AX,\xi)A\xi &= g(AX, JN)AJN = g(X, AJN)AJN \\ &= g(X, JAN)JAN = g(X, JN)JN \\ &= \eta(X)\xi. \end{aligned}$$

On the other hand, we can put

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$. So it becomes always $AY = BY$ for any vector field Y on M in Q^m. Then by differentiating (5.1) along any direction X, we have

$$\begin{aligned} (\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= 2(\nabla_X \eta)(Y) + 2\eta(Y)\nabla_X \xi + (\nabla_X B)Y. \end{aligned} \tag{5.2}$$

Suppose that the normal Jacobi operator $\bar{R}_N$ is parallel on M. Then (5.2) becomes

$$0 = 2g(\phi SX, Y)\xi + 2\eta(Y)\phi SX + (\nabla_X B)Y.$$

From this, putting $Y = \xi$, it becomes

$$2\phi SX = -(\nabla_X B)\xi = -(\nabla_X A)\xi. \tag{5.3}$$

On the other hand, differentiating the tangent vector $A\xi = B\xi$ and using the equation of Gauss and Weingarten formula, we have the following

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X \xi \\ &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi) - A\phi SX \\ &= (\bar{\nabla}_X A)\xi + A\bar{\nabla}_X \xi - \sigma(X, A\xi) - A\phi SX \\ &= q(X)JA\xi + A\phi SX - \sigma(X, A\xi) - A\phi SX + \eta(SX)AN, \end{aligned} \tag{5.4}$$

where in the final equality we have used $AN = N$ and $\bar{\nabla}_X \xi = \nabla_X \xi + g(SX, \xi)N$ and the Gauss equation $\bar{\nabla}_X(A\xi) = \nabla_X(A\xi) + \sigma(X, A\xi)$. From this, together with (5.3) and $AN = N$, we have

$$\begin{aligned} 2\phi SX &= -\{-q(X)AN + A\phi SX - \sigma(X, A\xi) - A\phi SX + \alpha\eta(X)AN\} \\ &= \{q(X)N + \sigma(X, A\xi) - \alpha\eta(X)N\} \end{aligned} \tag{5.5}$$

for any tangent vector field X on M in Q^m, where we have used $JA\xi = -JAJN = J^2AN = -AN$. By comparing the normal and tangential part of (5.5) respectively, we have

$$\phi SX = 0$$

for any vector field X on M. This gives $SX = \eta(SX)\xi$, that is, M is said to be totally η-umbilical, which implies $S\phi X = \phi SX$ for any vector field X on M in Q^m. Then by Theorem 1.1 due to Suh [15] and [16], M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in a complex quadric Q^{2k}. But they are never totally η-umbilical. So we conclude that there does not exist any real hypersurfaces in complex quadric Q^m with parallel normal Jacobi operator when the unit normal vector field N is $\mathfrak{A}$-principal.

6 Parallel normal Jacobi operator with $\mathfrak{A}$-isotropic normal

In this section let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be put

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (3.3) and the anti-commuting $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0.$$

By virtue of these formulas for an $\mathfrak{A}$-isotropic unit normal, the normal Jacobi operator $\bar{R}_N$ in Section 4 is given by

$$\bar{R}_N(Y) = Y + 3\eta(Y)\xi - g(AY, N)AN - g(AY, \xi)A\xi.$$

Now let us assume that the normal Jacobi operator $\bar{R}_N$ on M is parallel. Then it gives that

$$\begin{aligned} 0 =\ & (\nabla_X \bar{R}_N)Y \\ =\ & 3(\nabla_X\eta)(Y)\xi + 3\eta(Y)\nabla_X\xi - g(\nabla_X(AN), Y)AN \\ & -g(AN, Y)\nabla_X(AN) - g(Y, \nabla_X(A\xi))A\xi - g(A\xi, Y)\nabla_X(A\xi). \end{aligned} \tag{6.1}$$

On the other hand, by using the equation of Gauss we know that

$$\begin{aligned} \nabla_X(AN) =\ & \bar{\nabla}_X(AN) - \sigma(X, AN) \\ =\ & (\bar{\nabla}_X A)N + A\bar{\nabla}_X N - \sigma(X, AN) \\ =\ & q(X)JAN - ASX - \sigma(X, AN), \\ =\ & q(X)A\xi - ASX - \sigma(X, AN), \end{aligned}$$

and

$$\begin{aligned} \nabla_X(A\xi) =\ & \bar{\nabla}_X(A\xi) - \sigma(X, A\xi) \\ =\ & (\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi - \sigma(X, A\xi) \\ =\ & q(X)JA\xi + A\{\phi SX + \eta(SX)N\} - \sigma(X, A\xi) \\ =\ & -q(X)AN + A\phi SX + \eta(SX)AN - \sigma(X, A\xi). \end{aligned}$$

Substituting these formulas into (6.1) and putting $Y = \xi$ in the obtained equation, we know

$$\begin{aligned} 0 =\ & 3(\nabla_X\eta)(\xi)\xi + 3\nabla_X\xi - g(\nabla_X(AN), \xi)AN - g(\xi, \nabla_X(A\xi))A\xi \\ =\ & 3\phi SX + g(AN, \phi SX)AN + g(\phi SX, A\xi)A\xi. \end{aligned} \tag{6.2}$$

Then by taking the inner product of (6.2) with AN and using $g(AN, AN) = 1$, and $g(AN, A\xi) = 0$, we have

$$g(\phi SX, AN) = 0.$$

Also, by applying $A\xi$ to (6.2) and using $g(A\xi, A\xi) = 1$, and $g(AN, A\xi) = 0$, it follows that

$$g(\phi SX, A\xi) = 0.$$

From these, together with (6.2), it follows that $\phi SX = 0$. This implies that $SX = \alpha\eta(X)\xi$, that is, M is totally η-umbilical. Then the shape operator S commutes with the structure tensor ϕ. Then by Theorem 1.1 in Suh [15], [16], M is locally congruent to a tube over a totally geodesic complex submanifold $\mathbb{C}P^k$ in Q^{2k}. But this kind of tube is never totally η-umbilical. Accordingly, we conclude that any real hypersurfaces M in Q^m with $\mathfrak{A}$-isotropic do not admit any parallel normal Jacobi operator.

Remark 6.1 When a real hypersurface M in Q^{2k} is locally congruent to a tube of radius r $(0 < r < \frac{\pi}{2})$ over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}, in Suh [15] and [17], we have introduced that the shape operator S commutes with the structure tensor ϕ. Moreover, it is known that the normal vector field N is $\mathfrak{A}$-isotropic and Hopf, that is, $S\xi = \alpha\xi$ and the Reeb function α is constant. If we suppose that the normal Jacobi operator is parallel, then by (5.2) the shape operator becomes $SX = \alpha\eta(X)\xi$, which is said to be totally η-umbilical. But, by virtue of the principal curvature given in [15], [16] the tube mentioned above is not totally η-umbilical. This means that the tube does not admit parallel normal Jacobi operator.

Remark 6.2 When we consider that M is locally congruent to a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, over a totally geodesic and totally real space form S^m in Q^m. Then in Suh [16] and [17] it is known that M has three distinct constant principal curvatures $\alpha = -\sqrt{2}\cot(\sqrt{2}r)$, $\lambda = 0$ and $\mu = \sqrt{2}\tan(\sqrt{2}r)$ with multiplicities 1, $m-1$ and $m-1$ respectively. This is equivalent to $\phi S + S\phi = k\phi$, where k is a constant $k \neq 0$. Moreover, the unit normal N of M in Q^m is $\mathfrak{A}$-principal, that is, $AN = N$, and $A\xi = -\xi$. If we assume that the normal Jacobi operator on M is parallel, then by (6.2) we know $\phi SX = 0$ for any vector field X on M, which gives that $SX = \alpha\eta(X)\xi$. Then $S\phi X = \alpha\eta(\phi X)\xi = 0$. From this, together with the above formula $\phi S + S\phi = k\phi$, it gives $k\phi = 0$, $k \neq 0$ const, which gives a contradiction. Accordingly, the tube mentioned above also does not admit parallel normal Jacobi operator.

Remark 6.3 In [17] we have classified real hypersurfacees M in complex quadric Q^m with parallel Ricci tensor, according to the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic normal. When N is $\mathfrak{A}$-principal, we proved a non-existence property for Hopf hypersurfaces in Q^m. For a Hopf real hypersurface M in Q^m with $\mathfrak{A}$-isotropic we have given a complete classification that M has *three distinct constant* principal curvatures.

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